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A Method for Solving Linear Fractional Differential Equations Involving Mixed Partial Derivatives

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Abstract:- This study presents the Laplace substitution method, which is a unique technique for finding exact or approximate solutions to linear fractional differential equations with mixed partial derivatives. This approach is simple, practical, and very successful for solving such problems. It provides a dependable foundation for tackling intricate problems with fractional derivatives and mixed partial terms. We provide three examples to show its usefulness and efficiency of the method. The findings demonstrate the method's simplicity and accuracy, making it a useful tool for solving linear fractional differential equations with mixed partial derivatives.

Keywords: Linear Fractional Differential Equation, Mixed Partial Fractional Differential equations, Laplace Substitution Method

1. Introduction

The study of fractional calculus has received much interest from researchers and applied mathematicians because of its wide range of applications in domains such as mathematics, science, engineering, plasma physics, material mechanics, biology, p, finance, and chemistry [1-6]. Analytical and numerical approaches such as the Laplace transform, Fourier transform, fractional Dirac operator [7], and Elzaki transform [8] have been used to solve linear fractional differential equations. Iterative approaches for nonlinear equations, including the Adomian decomposition and variational iterative methods, are successful. Because many fractional-order partial differential equations do not have accurate analytical solutions, approximation and numerical procedures, such as the fractional complex transformation [9], Homotopy perturbation method [10], and generalised differential transform method [14], are widely utilised.

This work offers the Laplace Substitution Method, a unique and fast approach for solving linear fractional differential equations with mixed partial derivatives. Dr. S. S. Handibag and Dr. B.

D. Karande [16] developed this approach for partial differential equations. Still, it has succeeded for both linear and nonlinear equations, including higher-order and integrodifferential forms. Its simplicity, low computing effort, and accuracy make it a viable alternative to current approaches.

The work is organised as follows: Section 2 introduces the essential concepts and theorems needed for the research. Section 3 describes the Laplace Substitution Method for linear fractional differential equations with mixed partial derivatives. Section 4 uses examples to explain how the approach may be applied. Finally, Section 5 concludes the paper with a summary of findings and remarks.

2. Basic definitions

A large amount of literature is available on different definitions of fractional derivatives. The following section describes fractional calculus theory's definitions, theorems, and characteristics [17].

Definition 2.1: For the case of Riemann-Liouville, we have the following definition:

$$D_x^{\alpha}\big(f(x)\big) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}\, f(t)dt \qquad (2.1)$$

where $'\Gamma'$ denotes gamma function, which is Mellin transform of exponential Function and is defined as

$$\Gamma y = \int_o^{\propto} t^{y-1}\, e^{-t}\, \mathrm{dt} \qquad \mathrm{Re}\,[y] > O. \qquad (2.2)$$

Definition 2.2: The fractional order derivative of Function x^{β}, $\beta >$ -1 is given as,

$$D_\chi^{\propto}\big(x^{\beta}\big) = \frac{x^{-\propto+\beta}\,\Gamma(1+\beta)}{\Gamma(1-\propto+\beta)} \qquad (2.3)$$

Definition 2.3: The Laplace transform of Fractional R-L derivative is

$$L\{D_\chi^{\propto}\,(F\,(x))\} = s^{\propto}\mathrm{F}\,(\mathrm{S}) - \sum_{k=0}^{n-1} s^k D_\chi^{\propto-k-1}(0) \quad , n\text{-}1<\propto\, \le n, \qquad (2.4)$$

where $\mathrm{F(S)} = L\{(F\,(x)\} = \int_o^{\propto} e^{-st}F\,(t)dt$

Theorem: Let f, g be α–differentiable at a point $t > 0$.

$$D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g), \text{ for all } a, b \in \mathrm{R} \qquad (2.5)$$

3. Laplace Substitution method:

The general form of linear fractional differential equations involving mixed partial derivatives with initial conditions is below.

$$\mathrm{Lu}\,(x,t) + \mathrm{Ru}\,(x,t) = \mathrm{h}\,(x,t) \qquad (3.1)$$

$$D_\chi^{\propto-1}(0,t) = c_1,\ D_\chi^{\propto-2}\,(0,t) = c_2, \ldots, D_\chi^{\propto-n}\,(0,t)\,) = c_n \qquad (3.2)$$

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& $D_t^{\beta-1}(x,0) = b_1$, $D_t^{\beta-2}(x,0) = b_2, \ldots, D_x^{\beta-n}(x,0) = b_n$ (3.3)

where $L = \frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\,\partial t^{\beta}}$, Ru (x,t) is a group of remaining linear terms and h(x,t) is the source term.

[n-1<∝ ≤ n & n-1< β ≤ n]

We can write (1) in the following form,

$$\frac{\partial^{\alpha+\beta} u(x,t)}{\partial x^{\alpha}\,\partial t^{\beta}} + \mathrm{Ru}(x,t) = \mathrm{h}(x,t)$$

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}}\right) + \mathrm{Ru}(x,t) = \mathrm{h}(x,t) \quad (3.4)$$

Substituting $\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = U(x,t)$ in (4), we get,

$$\frac{\partial^{\alpha} U(x,t)}{\partial x^{\alpha}} + \mathrm{Ru}(x,t) = \mathrm{h}(x,t) \quad (3.5)$$

Taking Laplace transform of (5) w. r. t. x, we get,

$$s^{\alpha}U(s,t) - s^{0}D_{\chi}^{\alpha-1}(0,t) - sD_{\chi}^{\alpha-2}(0,t) \ldots\ldots - s^{n-1}D_{\chi}^{\alpha-n}(0,t) = L_x\{h(x,t) - Ru(x,t)\} \quad (3.6)$$

From (2) $D_{\chi}^{\alpha-1}(0,t) = c_1$, $D_{\chi}^{\alpha-n}(0,t) = c_2$, ______ $D_{\chi}^{\alpha-n}(0,t) = c_n$

Equation (6) becomes,

$$s^{\alpha}U(s,t) - c_1 - sc_2 - \cdots\ldots - s^{n-1}c_n = L_x\{h(x,t) - Ru(x,t)\}$$

$$s^{\alpha}U(s,t) = c_1 + s + \cdots\ldots + s^{n-1}c_n + L_x\{h(x,t) - Ru(x,t)\}$$

$$U(s,t) = \frac{C_1}{s^{\alpha}} + \frac{C_2}{s^{\alpha-1}} + \ldots\ldots + \frac{C_n}{s^{\alpha-n+1}} + \frac{1}{s^{\alpha}} L_x\{h(x,t) - Ru(x,t)\}$$

Taking Inverse Laplace transform w. r. t. *x* on both sides, we get,

$$U(x,t) = L_x^{-1}\{\frac{C_1}{s^{\alpha}} + \frac{C_2}{s^{\alpha-1}} + \ldots\ldots + \frac{C_n}{s^{\alpha-n+1}} + \frac{1}{s^{\alpha}} L_x\{h(x,t) - Ru(x,t)\}$$

$$U(x,t) = \frac{C_1 x^{\alpha-1}}{\Gamma\alpha} + \frac{C_2 x^{\alpha-2}}{\Gamma(\alpha-1)} + \ldots\ldots + \frac{C_n x^{\alpha-n}}{\Gamma(\alpha-n+1)} + L_x^{-1}\left\{\frac{1}{s^{\alpha}} L_x\{h(x,t) - Ru(x,t)\}\right\} \quad (3.7)$$

But $U(x,t) = \frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}}$,

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = \frac{C_1 x^{\alpha-1}}{\Gamma\alpha} + \frac{C_2 x^{\alpha-2}}{\Gamma(\alpha-1)} + \ldots\ldots + \frac{C_n x^{\alpha-n}}{\Gamma(\alpha-n+1)} + L_x^{-1}\left\{\frac{1}{s^{\alpha}} L_x\{h(x,t) - Ru(x,t)\}\right\} \quad (3.8)$$

Taking Laplace transform of (8) w. r. t. *t* on both sides, we get,

$$s^{\beta}\mathrm{u}(x,s) - \sum_{k=0}^{n-1} S^{k} D_t^{\beta-k-1}(x,0) = L_t\left\{\frac{C_1 x^{\alpha-1}}{\Gamma\alpha} + \frac{C_2 x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots\ldots + \frac{C_n x^{\alpha-n}}{\Gamma(\alpha-n+1)} + L_x^{-1}\left\{\frac{1}{s^{\alpha}} L_x\{h(x,t) - Ru(x,t)\}\right\}\right\}$$

$$s^{\beta}u\,(x,s) - s^0 D_\chi^{\beta-1}(x,0) - sD_\chi^{\beta-2}(x,0) \ldots.. - s^{n-1} D_\chi^{\beta-n}(x,0) = L_t\left\{\frac{c_1 x^{\propto-1}}{\Gamma\alpha} + \frac{c_2 x^{\propto-2}}{\Gamma(\alpha-1)} + \cdots..+ \frac{c_n x^{\propto-n}}{\Gamma(\alpha-n)} + L_x^{-1}\left\{\frac{1}{s^\alpha} L_x\{h\,(x,t) - Ru\,(x,t)\}\right\}\right\} \quad (3.9)$$

From (3) $D_\chi^{\beta-1}(x,0) = b_1$, $D_t^{\beta-2}(x,0) = b_2$, ______ $D_t^{\beta-n}(x,0) = b_n$

Equation (9) becomes,

$$s^{\beta}u(x,s) - b_1 - sb_2 \ldots.. - s^{n-1}b_n = L_t\left\{\frac{c_1 x^{\propto-1}}{\Gamma\alpha} + \frac{c_2 x^{\propto-2}}{\Gamma(\alpha-1)} + \cdots..+ \frac{c_n x^{\propto-n}}{\Gamma(\alpha-n+1)} + L_x^{-1}\left\{\frac{1}{s^\alpha} L_x\{h\,(x,t) - Ru\,(x,t)\}\right\}\right\}$$

$$s^{\beta}u(x,s) = b_1 + sb_2 \ldots..+ s^{n-1}b_n + L_t\left\{\frac{c_1 x^{\propto-1}}{\Gamma\alpha}\frac{c_2 x^{\propto-2}}{\Gamma(\alpha-1)} + \ldots.. + \frac{c_n x^{\propto-n}}{\Gamma(\alpha-n+1)} + L_x^{-1}\left\{\frac{1}{s^\alpha} L_x\{h\,(x,t) - Ru\,(x,t)\}\right\}\right\}$$

$$s^{\beta}u(x,s) = b_1 + sb_2 \ldots..+ s^{n-1}b_n + + \frac{1}{s}\left[\frac{c_1 x^{\propto-1}}{\Gamma\alpha}\frac{c_2 x^{\propto-2}}{\Gamma(\alpha-1)} + \ldots.. + \frac{c_n x^{\propto-n}}{\Gamma(\alpha-n+1)}\right] + L_t\left\{L_x^{-1}\left\{\frac{1}{s^\alpha} L_x\{h\,(x,t) - Ru\,(x,t)\}\right\}\right\} \quad (3.10)$$

Taking Inverse Laplace transform w. r. t. t of (9) on both sides, we get,

$$u(x,t) = L_t^{-1}\left\{\frac{b_1}{s^\beta} + \frac{b_2}{s^{\beta-1}} + \frac{b_2}{s^{\beta-n+1}} + \frac{1}{s^{\beta+1}}\left[\frac{c_1 x^{\propto-1}}{\Gamma\alpha}\frac{c_2 x^{\propto-2}}{\Gamma(\alpha-1)} + \ldots.. + \frac{c_n x^{\propto-n}}{\Gamma(\alpha-n+1)}\right] + L_t\left\{L_x^{-1}\left\{\frac{1}{s^\alpha} L_x\{h\,(x,t) - Ru\,(x,t)\}\right\}\right\}\right\}$$

$$U(x,t) = \frac{b_1 t^{\beta-1}}{\Gamma\beta} + \frac{b_2 t^{\beta-2}}{\Gamma(\beta-1)} + \ldots..+ \frac{b_n t^{\beta-n}}{\Gamma(\beta-n+1)} + \frac{t^\beta}{\Gamma(\beta-1)}\left[\frac{c_1 x^{\propto-1}}{\Gamma\alpha}\frac{c_2 x^{\propto-2}}{\Gamma(\alpha-1)} + \ldots.. + \frac{c_n x^{\propto-n}}{\Gamma(\alpha-n+1)}\right] + L_t^{-1}\left\{\frac{1}{s^\beta} L_t\left\{L_x^{-1}\left\{\frac{1}{s^\alpha} L_x\,\{h\,(x,t) - Ru\,(x,t)\}\right\}\right\}\right\}$$

is the required solution of equation (1).

4. Applications:

Example 1: $\frac{\partial^{\propto+\beta} u\,(x,t)}{\partial x^{\propto}\,\partial t^{\beta}} = 0 \quad (4.1)$

with initial conditions

$D_x^{\propto-1}(0,t) = c_1$, $D_x^{\propto-2}(0,t) = c_2, \ldots..,$ $D_x^{\propto-n}(0,t) = c_n$,

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and $D_t^{\beta-1}(x,0) = b_1, D_t^{\beta-2}(x,0) = b_2, \ldots, D_t^{\beta-n}(x,0) = b_n$,

where c_i is either constant or function of t and b_i is either constant or function of x

Let us assume

$$\frac{\partial^\beta u}{\partial t^\beta} = U \Rightarrow \frac{\partial^\alpha U}{\partial x^\alpha} = 0 \quad (4.2)$$

which is a homogeneous fractional differential equation.

Taking Laplace transform on both sides of equation (4.2) w. r. t. x,

$$s^\alpha U(s,t) - \sum_{k=0}^{n-1} s^k D_x^{\alpha-k-1}(0,t) = 0$$

$$U(s,t) = \frac{c_1}{s^\alpha} + \frac{c_2}{s^{\alpha-1}} + \ldots + \frac{c_n}{s^{\alpha-n+1}} \quad (4.3)$$

Taking inverse Laplace transform on both sides of equation (4.3) w. r. t. x,

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{c_1 x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2 x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots + \frac{c_n x^{\alpha-n}}{\Gamma(\alpha-n+1)} \quad (4.4)$$

Taking Laplace transform on both sides of equation (4.4) w. r. t. t,

$$s^\beta u(x,s) - \sum_{k=0}^{n-1} s^k D_t^{\beta-k-1}(x,0) = L_t\left\{\frac{c_1 x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2 x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots + \frac{c_n x^{\alpha-n}}{\Gamma(\alpha-n+1)}\right\}$$

$$s^\beta u(x,s) - b_1 - s b_2 - \cdots - s^{n-1} b_n = \frac{1}{S}\left\{\frac{c_1 x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2 x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots + \frac{c_n x^{\alpha-n}}{\Gamma(\alpha-n+1)}\right\}$$

$$u(x,s) = \frac{b_1}{s^\beta} + \frac{b_2}{s^{\beta-1}} + \ldots + \frac{b_n}{s^{\beta-n+1}} + \frac{1}{s^{\beta+1}}\left\{\frac{c_1 x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2 x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots + \frac{c_n x^{\alpha-n+1}}{\Gamma(\alpha-n)}\right\}$$

Taking inverse Laplace transform on both sides of the above equation w. r. t. t,

$$u(x,t) = \frac{b_1 t^{\beta-1}}{\Gamma\beta} + \frac{b_2 t^{\beta-2}}{\Gamma(\beta-1)} + \cdots + \frac{b_n t^{\beta-n}}{\Gamma(\beta-n+1)} + \frac{t^\beta}{\Gamma(\beta+1)}\left\{\frac{c_1 x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2 x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots + \frac{c_n x^{\alpha-n}}{\Gamma(\alpha-n+1)}\right\} \quad (4.5)$$

The above equation (4.5) solves the problem (4.1).

Case I] – $c_i = 0, b_i = 0, 1 \le i \le n$

$u(x,t) = 0$ which is a trivial solution.

Case II] – $c_i = 0, b_i = 1, 1 \le i \le n$

$u(x,t) = \frac{t^{\beta-1}}{\Gamma\beta} + \frac{t^{\beta-2}}{\Gamma(\beta-1)} + \cdots + \frac{t^{\beta-n}}{\Gamma(\beta-n+1)} = \sum_{k=1}^{n} \frac{t^{\beta-k}}{\Gamma(\beta-k+1)}$. The graphical presentation of (4.5) for α = 1.3 and β = 1.6 is in Fig 1.

Case III] – $c_i = t^i, b_i = 0, 1 \le i \le n$

$u(x,t) = \frac{t^\beta}{\Gamma(\beta+1)} \sum_{k=1}^{n} \frac{t^k x^{\alpha-k}}{\Gamma(\alpha-k+1)}$. The graphical presentation of (4.5) for α = 1.3 and β = 1.6 is given in Fig 2.

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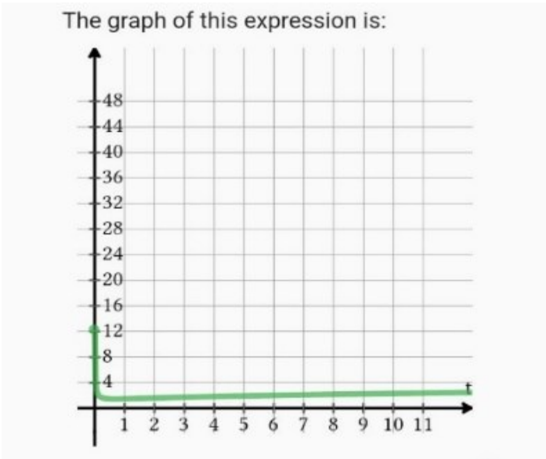

Fig:1 For α = 1.3 and β = 1.6

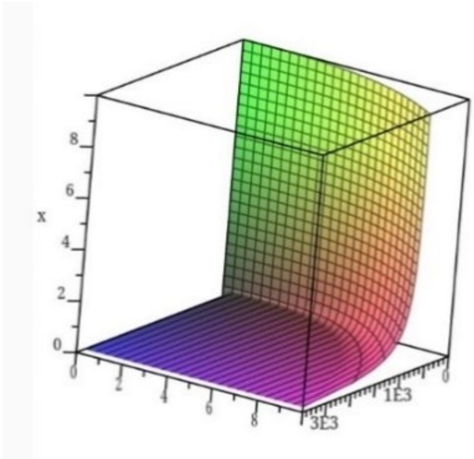

Fig:2 For α = 1.3 and β = 1.6

Example 2:

$$\frac{\partial^{\alpha+\beta} u(x,t)}{\partial x^{\alpha}\,\partial t^{\beta}} = x \tag{4.6}$$

with initial conditions, $D_x^{\alpha-1}(0,t) = c_1,\ D_x^{\alpha-2}(0,t) = c_2, \ldots.., D_x^{\alpha-n}(0,t) = c_n,$

and $D_t^{\beta-1}(x,0) = b_1, D_t^{\beta-2}(x,0) = b_2, \ldots.., D_t^{\beta-n}(x,0) = b_n,$

where c_i is either constant or a function of t and b_i is either constant or a function of x

Let us assume

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} = U \Rightarrow \frac{\partial^{\alpha} U}{\partial x^{\alpha}} = x \tag{4.7}$$

which is a non-homogeneous fractional differential equation.

Taking Laplace transform on both sides of equation (2) w. r. t. x,

$$s^{\alpha}U(s,t) - \sum_{k=0}^{n-1} s^k D_x^{\alpha-k-1}(0,t) = \frac{1}{s^2}$$

$$U(s,t) = \frac{c_1}{s^{\alpha}} + \frac{c_2}{s^{\alpha-1}} + \ldots. + \frac{c_n}{s^{\alpha-n+1}} + \frac{1}{s^{\alpha+2}} \tag{4.8}$$

Taking inverse Laplace transform on both sides of equation (3) w. r. t. x,

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = \frac{c_1 x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2 x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots. + \frac{c_n x^{\alpha-n}}{\Gamma(\alpha-n+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} \tag{4.9}$$

Taking Laplace transform on both sides of equation (4) w. r. t. t,

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$$s^{\beta}u\ (x,s) - \sum_{k=o}^{n-1} s^{k}\, D_{t}^{\beta-k-1}(x,0) = L_{t}\left\{\frac{c_1\, x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2\, x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots . + \frac{c_n\, x^{\alpha-n}}{\Gamma(\alpha-n+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right\}$$

$$s^{\beta}u\ (x,s) - b_1 - sb_2 - \cdots - s^{n-1}b_n$$

$$= \frac{1}{s}\left\{\frac{c_1\, x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2\, x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots . + \frac{c_n\, x^{\alpha-n}}{\Gamma(\alpha-n+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right\}$$

$$u\ (x,s) = \frac{b_1}{s^{\beta}} + \frac{b_2}{s^{\beta-1}} + \ldots .. + \frac{b_n}{s^{\beta-n+1}} + \frac{1}{s^{\beta+1}}\left\{\frac{c_1\, x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2\, x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots . + \frac{c_n\, x^{\alpha-n}}{\Gamma(\alpha-n+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right\}$$

Taking inverse Laplace transform on both sides of equation (5) w. r. t. *t*,

$$u\ (x,t) = \frac{b_1\, t^{\beta-1}}{\Gamma\beta} + \frac{b_2\, t^{\beta-2}}{\Gamma(\beta-1)} + \ldots + \frac{b_n\, t^{\beta-n}}{\Gamma(\beta-n+1)} + \frac{t^{\beta}}{\Gamma(\beta+1)}\left\{\frac{c_1\, x^{\alpha-1}}{\Gamma\alpha}\ \frac{c_2\, x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots + \frac{c_n\, x^{\alpha-n}}{\Gamma(\alpha-n+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right\} \tag{4.10}$$

The equation (4.10) solves the given mixed-order partial fractional differential equation problem (4.6).

Case I] – $c_i = 0, b_i = 0, 1 \le i \le n$

$u\ (x,t) = \frac{t^{\beta}x^{\alpha+1}}{\Gamma(\beta+1)\Gamma(\alpha+2)}$. The graphical presentation of (4.10) for α = 1.3 and β = 1.6 is given in Fig 3.

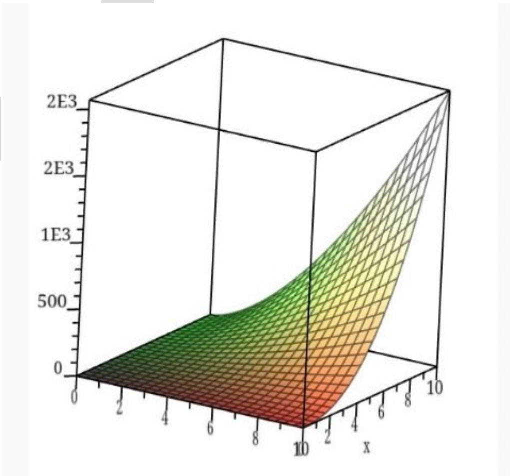

Fig:3 For α = 1.3 and β = 1.6

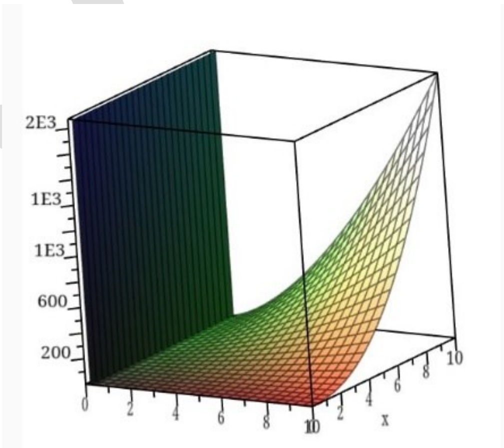

Fig:4 For α = 1.3 and β = 1.6

Case II] – $c_i = 0, b_i = 1, 1 \le i \le n$

$$u\ (x,t) = \frac{t^{\beta}x^{\alpha+1}}{\Gamma(\beta+1)\Gamma(\alpha+2)} \sum_{k=1}^{n} \frac{t^{\beta-k}}{\Gamma(\beta-k+1)}.$$ The graphical presentation is in Fig 4.

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Case III] $c_i = t^i, b_i = 0, 1 \le i \le n$

$u\,(x,t) = \frac{t^{\beta} x^{\alpha+1}}{\Gamma(\beta+1)\Gamma(\alpha+2)} + \sum_{k=1}^{n} \frac{t^k x^{\alpha-k}}{\Gamma(\alpha-k+1)}.$

The graphical presentation of (4.10) for α = 1.3 and β = 1.6 is given in Fig 5.

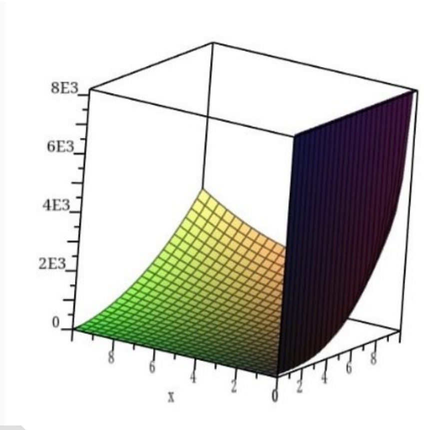

Fig:5 For α = 1.3 and β = 1.6

Example 3:

$$\frac{\partial^{\alpha+\beta} u\,(x,t)}{\partial x^{\alpha}\, \partial t^{\beta}} = x^2\, t \quad (4.11)$$

With initial conditions

$D_x^{\alpha-1}(0,t) = c_1, \quad D_x^{\alpha-2}(0,t) = c_2, \ldots., \quad D_x^{\alpha-n}(0,t) = c_n,$

and $D_t^{\beta-1}(x,0) = b_1, D_t^{\beta-2}(x,0) = b_2, \ldots.., D_t^{\beta-n}(x,0) = b_n$

Let us assume

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} = \mathrm{U} \Rightarrow \frac{\partial^{\alpha} U}{\partial x^{\alpha}} = x^2\, t \quad (4.12)$$

which is a non-homogeneous fractional differential equation.

Taking Laplace transform on both sides of equation (2) w. r. t. x,

$s^{\alpha}\mathrm{U}\,(s,t) - \sum_{k=o}^{n-1} s^k\, D_x^{\alpha-k-1}(o,t) = \frac{2!t}{s^3}$

$$\mathrm{U}(s,t) = \frac{2t}{s^{\alpha+3}} + \frac{c_1}{s^{\alpha}} + \frac{c_2}{s^{\alpha-1}} + \ldots.+\frac{c_n}{s^{\alpha-n+1}} \quad (4.13)$$

Taking inverse Laplace transform on both sides of equation (3) w. r. t. x,

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = \frac{2tx^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{c_1\, x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2\, x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots.+\frac{c_n\, x^{\alpha-n}}{\Gamma(\alpha-n+1)} \quad (4.14)$$

Taking Laplace transform on both sides of equation (4) w. r. t. t,

$$s^{\beta}u\,(x,s) - \sum_{k=o}^{n-1} s^k\, D_t^{\beta-k-1}(x,0) = \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)}\frac{1}{s^2} + L_t\left\{\frac{c_1\, x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2\, x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots.+\frac{c_n\, x^{\alpha-n}}{\Gamma(\alpha-n+1)}\right\}$$

$$u\,(x,s) = \frac{b_1}{s^{\beta}} + \frac{b_2}{s^{\beta-1}} + \ldots..+\frac{b_n}{s^{\beta-n+1}} + \frac{2x^{\alpha+2}}{s^{\beta+2}\Gamma(\alpha+3)} + \frac{1}{s^{\beta+1}}\left\{\frac{c_1\, x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2\, x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots.+\frac{c_n\, x^{\alpha-n+1}}{\Gamma(\alpha-n)}\right\} \quad (4.15)$$

Taking inverse Laplace transform on both sides of equation (5) w. r. t. *t*,

$$u\,(x,t) = \frac{b_1\,t^{\beta-1}}{\Gamma\beta} + \frac{b_2\,t^{\beta-2}}{\Gamma(\beta-1)} + \ldots + \frac{b_n\,t^{\beta-n}}{\Gamma(\beta-n+1)} + \frac{2x^{\alpha+2}t^{\beta+1}}{\Gamma(\beta+2)\Gamma(\alpha+3)} + \frac{t^{\beta}}{\Gamma(\beta+1)}\left\{\frac{c_1\,x^{\alpha-1}}{\Gamma\alpha} + \frac{c_2\,x^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots + \frac{c_n\,x^{\alpha-n}}{\Gamma(\alpha-n+1)}\right\} \quad (4.16)$$

The above equation is the solution of the problem (4.11).

Case I] – $c_i = 0, b_i = 0, 1 \le i \le n$

$u\,(x,t) = \frac{2t^{\beta+1}x^{\alpha+2}}{\Gamma(\beta+2)\Gamma(\alpha+3)}$. The graphical presentation of the surface (4.16) For α = 1.3 and β = 1.6 is in Fig 6.

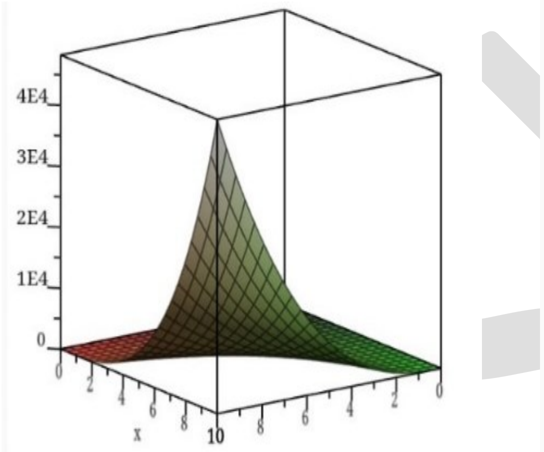

Fig:6 For α = 1.3 and β = 1.6

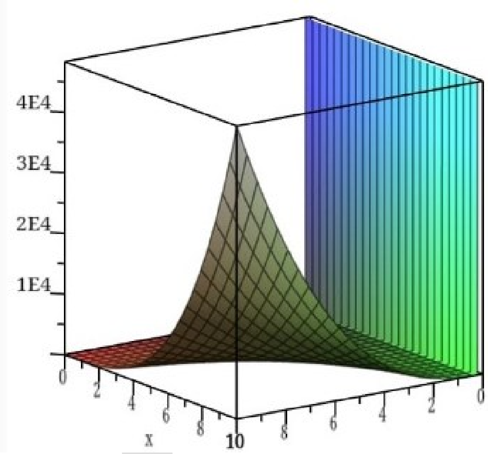

Fig:7 For α = 1.3 and β = 1.6

Case II] – $c_i = 0, b_i = 1, 1 \le i \le n$

$u\,(x,t) = \frac{2t^{\beta+1}x^{\alpha+2}}{\Gamma(\beta+2)\Gamma(\alpha+3)} + \sum_{k=1}^{n}\frac{t^{\beta-k}}{\Gamma(\beta-k+1)}$. The graphical presentation of the surface (4.16) for α = 1.3 and β = 1.6 is in Fig 7.

Case III] – $c_i = t^i, b_i = 0, 1 \le i \le n$

$u\,(x,t) = \frac{2t^{\beta+1}x^{\alpha+2}}{\Gamma(\beta+2)\Gamma(\alpha+3)} + \frac{t^{\beta}}{\Gamma(\beta+1)}\sum_{k=1}^{n}\frac{t^{k}x^{\alpha-k}}{\Gamma(\alpha-k+1)}$.

The graphical presentation of the surface (4.16) for α = 1.3 and β = 1.6 is in Fig 8.

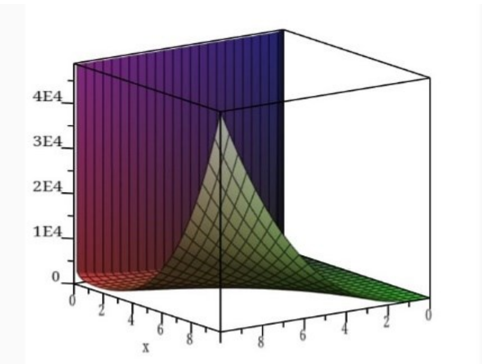

Fig:8 For α = 1.3 and β = 1.6

5. Conclusion:

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This study illustrates the usefulness of the Laplace Substitution Method for solving linear multivariate fractional differential equations with mixed partial derivatives, especially if the general linear components are zero, i.e., Ru (x, t) = 0. The study demonstrates that this technique yields precise and accurate approaches to both homogeneous and non-homogeneous linear fractional differential equations. In Example 1, the approach is used to a homogeneous linear fractional differential equation with mixed partial derivatives to demonstrate its ease of use and effectiveness.

Examples 2 and 3 further validate the method's applicability to non-homogeneous equations, showcasing its versatility in handling various problems. To support the findings, graphical representations of the solutions are presented for specific cases using Maple Software, providing visual confirmation of the method's accuracy. The Laplace Substitution Method stands out for its simplicity, requiring fewer calculations compared to other methods while still delivering exact solutions. It proves to be a reliable and effective tool for solving complex linear fractional differential equations involving mixed partial derivatives, making it an invaluable approach in applied mathematics and engineering problems.

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